

Shareholding interlocks: profit formulations and cartelizing effects

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Abstract

Different profit formulations are considered when studying collusive effects of horizontal cross-shareholding. We propose a compact formula which generalizes the formulations used in the literature. By this compact formula we derive an expression for the Lerner index and discuss the cartelizing effects of shareholding interlocks. We provide a simple counterexample which allows to identify some inconsistencies of the approach commonly used in the literature. Finally we show how our approach can be extended to consider product differentiation.

Keywords: Shareholding interlocks, cartel, financial control.

1. Introduction

Different contributions approach the cartelizing effects of shareholding interlocks: situations in which rivals hold shares in one another. When profit functions of the firms in an industry are linked via cross-shareholdings, firms take account of the effects their actions may have on their competitors. This induces collusive behavior and cartelizing effects, in the sense that competition reduces. In [14] the Authors prove that the linking of profits reduces each firm’s incentive to compete and, in particular, they investigate the competitive effects posed by partial ownership arrangements. In [7] a profit formulation for indirect shareholding (i.e., when a firm holds shares in a different firm and that

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firm also holds shares in another firm) is introduced. The main result proves that, in industries with horizontal shareholding, cartelizing effects are compounded if firms are mindful also of indirect shareholding, and in this case the industry may become a monopoly.

In [4] this profit formulation is considered in order to study the Dutch financial sector by comparing the case of cross-shareholding to the case of no-shareholding, both in a Cournot and a Bertrand market. The Authors give evidence that empirical observations of shareholding interlocks differ from the prediction of the theoretical model but do not address explicitly this problem. Finally, it must be noted that while they realize that with the profit formulation introduced in [7] profit may be overcounted, their explanation for considering this particular profit formulation is not convincing.

This paper addresses the overcounting profit problem, proposing the use of a different profit formulation which first, considers correctly how profits are shared and second, solves the problems of the formulation commonly used in the literature. Furthermore we generalize the commonly used profit formulations and the proposed one in a single general expression. We derive an elegant expression for the Lerner index and discuss the approach used in the literature for studying the cartelizing effects; the mathematical analysis of some overlooked details allows to understand how this may undermine the theoretical results established in [7]. We provide a simple counterexample in order to show how optimal quantities may not always be defined and show that the fact that financial interest profitability depends on different marginal costs rather than holding per se, gives a simple motivation of the limited and non-symmetrical shareholding interlocks found in the empirical literature. In particular our approach distinguishes cartelizing effects of the small financial interests observed in the empirical literature ([4]) from the large financial interests studied in the theoretical literature ([6] and [1992]). Finally we are able to extend the compact formalization in order to

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1For example [4] observe that, with the formulation commonly used in the literature, the total profit exceeds the total operating earning of the industry. Yet, they do not mention that with such a formulation profits may become unbounded.

2[6] claims: "...empirical examples of cross-shareholding among rivals are almost nonexistent even in Japan where share interlocks between firms that are not rivals are quite common indeed."

3In this case no firm does control more than 17% in any rival, and for each firm total shares controlled by rivals are less than 32%.

4In the sense that in each firm the shares not controlled by rivals tend to be close to 0.
consider differentiated products and derive the relative expression for the Lerner index.

The structure of the paper is the following. In Section 2 we present the model, discuss the existing profit formulations and give a compact formula which generalizes them. In Section 3 we discuss the approach used in the literature to study the cartelizing effects and derive a common expression for the Lerner index. In Section 4 by a simple counterexample where we study the particular case of two firms with linear inverse demand function and linear costs, we show that some of the results known in the literature seem to be unsound. In Section 5 we generalize the matrix notation to the multiproduct oligopoly and, in the last Section, summarize our results and give conclusions.

2. The model and profit formulations

Consider a Cournot-type model of a homogeneous-product industry with \( n \) firms with silent financial interests (see [2]), i.e., each firm seeks to maximize the value of its profits, including returns on any shares held in rivals, but controls only its own output\(^5\). Let the firms be in a static Cournot-Nash equilibrium in which choices of quantities represent strategies, and assume that the conditions for stable Cournot equilibria hold globally.

Furthermore consider:

- \( \mathbf{q} := (q_1, q_2, \ldots, q_n) \): vector of output controlled by each firm,
- \( q := \sum_{i=1}^{n} q_i \): total quantity,
- \( p(q) \): market inverse demand function differentiable at least once with elasticity \( \xi(q) \neq 0 \) at the equilibrium point,
- \( c_i(q_i), i = 1, 2, \ldots, n \): Firm \( i \) cost function; all are differentiable at least once,
- \( \delta_{ki} \): \( k \)-th firm’s ownership interest in the \( i \)-th firm.

In [13] the following formalizations were considered:

\(^5\)In this paper we consider managers controlling firms; potential sources of profit, are shares of profit from firms in which managers have controlling interests, from ownership interests in joint venture firms they do not control and, filially, from non-controlling interests in other firms.
Formulation 1, (Joint ventures):

\[
\pi_i = \left(1 - \sum_{k \neq i} \delta_{ki}\right)\left[p(q)q_i - c(q_i)\right] + \sum_{k \neq i} \delta_{ik}\left[p(q)q_k - c(q_k)\right].
\] (1)

In this case managers of Firm \(i\) maximize profits net of those going to competitors; this formulation is appropriate in the case of joint venture and in the case in which managers are also owners. Furthermore managers are aware of rivals’ claims on their firm profit.

Formulation 2 (Partial equity interests):

\[
\pi_i = p(q)q_i - c(q_i) + \sum_{k \neq i} \delta_{ik}(p(q)q_k - c(q_k)).
\] (2)

In this case managers consider all stockholders alike and do not consider ownership by rivals. This formulation may be appropriate when these interests are small, and do not convey control.

[7] introduced the following

Formulation 3 (Indirect shareholding):

\[
\pi_i = (p(q)q_i - c(q_i)) + \sum_{k \neq i} \delta_{ik}\pi_k.
\] (3)

Each firm’s objective is to maximize its profit \(\pi_i\) which includes its operating earnings \(p(q)q_i - c(q_i)\) and its return on equity holding in the other firms \(\sum_{k \neq i} \delta_{ik}\pi_k\). In this case we have silent financial interests since here firms hold shares in firms also holding shares in other firms. This formulation is the most commonly used in the literature ([6], [7], [4] and [10]); in particular it has been used to prove cartelizing effects of shareholding interlocks.

We propose the analogous to joint venture formulation in the case of silent interests.

Formulation 4 (Net indirect shareholding):

\[
\pi_i = \left(1 - \sum_{k \neq i} \delta_{ki}\right)\left(p(q)q_i - c(q_i)\right) + \sum_{k \neq i} \delta_{ik}\pi_k^G.
\] (4)

where \(\pi_k^G\) is Firm \(i\)’s gross profit. Gross profits are, under some conditions\(^6\), implicitly defined by

\[
\pi_i^G = p(q)q_i - c(q_i) + \sum_{k \neq i} \delta_{ik}\pi_k^G.
\] (5)

\(^6\)Namely no firm must be completely controlled by rivals: \(\sum_{i \neq j} \delta_{ij} < 1, \forall j = 1, 2, \ldots, n\). See [10] for further details.
The formulation is similar to the previous one since we have silent interests but, analogously to Formulation 1, both operating earnings and equity holdings are netted. In this case managers are aware of rivals’ claims on their profit and these must be considered when firm shareholding is not small. Both [4] and [8] use a formulation which is similar to the proposed one to prove that profits are not overstated. Yet in deriving their results they use Formulation 3.

Taking into account these analogies between profit formulations they may be classified in the following way:

<table>
<thead>
<tr>
<th>Net Operating earning</th>
<th>Operating earning and equity holding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Formulation 1</td>
<td>Formulation 4</td>
</tr>
<tr>
<td>Formulation 2</td>
<td>Formulation 3</td>
</tr>
</tbody>
</table>

Proposition 1. All the formulations may be obtained from the following common compact form

$$\pi = A \cdot [P(q) \cdot q - C(q) \cdot 1]$$

where:

- $A$: square matrix opportunely defined for each formulation,
- $P(q) := p(q)I$, where $I$ is the order $n$ identity matrix,
- $C(q)$: diagonal matrix where $i$-th diagonal element is Firm $i$’s cost function,
- $1$: $n$ column vector with unitary components.

Proof. Define:

- $D$: shareholding matrix with typical elements $\delta_{ij}$ representing Firm $i$’s fractional shareholding in Firm $j$, where $\delta_{ij} \geq 0$ if $i \neq j$, and $\delta_{ij} = 0$ if $i = j$. We suppose that no firm is completely controlled by the other firms; this means: $\sum_{i \neq j} \delta_{ij} < 1$, $\forall j = 1, 2, \ldots, n$.
- $(1)$: $n$ dimensional square matrix with unitary components.

With the notation it is a question of simple algebra to obtain the different formulations when considering respectively:

- $A = \text{diag}[I - (1) \cdot D] + D$ for Formulation 1,
- $A = (I + D)$ for Formulation 2,
- $A = (I - D)^{-1}$ for Formulation 3,
$A = \text{diag}[I - (1) \cdot D] \cdot (I - D)^{-1}$. For Formulation 4. \hfill \Box$

It must be noted that in the last two formulations matrix $A$ is defined since we assume that no firm is completely controlled by other firms. Obviously, this way, it is possible (see [15]) to express $(I - D)^{-1}$ as a power series:

$$(I - D)^{-1} = I + D + D^2 + D^3 + \ldots .$$

This expansion, together with (6), allows the following decomposition:

$$\pi = \text{diag} [I - (1) \cdot D] \cdot [I + D + D^2 + D^3 + \ldots] \cdot [P(q) \cdot q - C(q) \cdot 1].$$

This way the vector of firms’ (net) profit consists of

- netted operative earning: $\text{diag} [I - (1) \cdot D] \cdot [P(q) \cdot q - C(q) \cdot 1]$,
- netted effect of direct shareholding\(^7\): $\text{diag} [I - (1) \cdot D] \cdot D \cdot [P(q) \cdot q - C(q) \cdot 1]$,
- netted effect of indirect shareholding\(^8\):
  $$\text{diag} [I - (1) \cdot D] \cdot [D^2 + D^3 + \ldots] \cdot [P(q) \cdot q - C(q) \cdot 1].$$

The main difference between Formulation 3 and Formulation 4 lies in how profit are counted, in fact:

**Proposition 2.** If operating earnings are non null, with Formulation 3, profits are overcounted and in the limit case become infinite. By contrast when considering Formulation 4 the aggregate net profit is the sum of operative earnings of all firms in the industry.

**Proof.** Examining expression 3 it is evident that, with Formulation 3, Firm $i$’s profit does not take into account the share of profits claimed by firms controlling Firm $i$; this way profits are overcounted since the same share of profit is accrued both to Firm $i$ and its controlling firms. The consequences of this are particularly evident even in the simple case of two firms ($n = 2$)

$$\pi_1 = (p(q_1)q_1 - c(q_1)) + \delta_{12}\pi_2,$$

$$\pi_2 = (p(q_2)q_2 - c(q_2)) + \delta_{21}\pi_1$$

\(^7\)Shares held directly.

\(^8\)Shares held via another firm.
and solving for profits:

\[ \pi_1 = \frac{(p(q)q_1 - c(q_1)) + \delta_{12}(p(q)q_2 - c(q_2))}{1 - \delta_{12}\delta_{21}} \]

\[ = \left[p(q)q_1 - c(q_1) + \delta_{12}[p(q)q_2 - c(q_2)]\right] \sum_{i=0}^{\infty} (\delta_{12}\delta_{21})^i \]

\[ \pi_2 = \frac{(p(q)q_2 - c(q_2)) + \delta_{21}(p(q)q_1 - c(q_1))}{1 - \delta_{12}\delta_{21}} \]

\[ = \left[p(q)q_2 - c(q_2) + \delta_{21}[p(q)q_1 - c(q_1)]\right] \sum_{i=0}^{\infty} (\delta_{12}\delta_{21})^i \]

Then, when \( \delta_{12} = \delta_{21} = \delta - 1 \), \( \pi_t \) tends to infinity.

For the second part, under Formulation 4 profits are:

\[ \pi = \text{diag}\left[ I - (1) \cdot D \right] \cdot (I - D)^{-1} \cdot \left[ P(q) \cdot q - C(q) \cdot 1 \right] \]

(7)

Summing up we obtain aggregated profits:

\[ 1 \cdot \pi = 1 \cdot \text{diag}\left[ I - (1) \cdot D \right] \cdot (I - D)^{-1} \cdot \left[ P(q) \cdot q - C(q) \cdot 1 \right] \]

Since it is immediate to prove that \( 1 \cdot (I - D) = 1 \cdot \text{diag}\left[ I - (1) \cdot D \right] \), we have:

\[ 1 \cdot \pi = 1 \cdot (I - D) \cdot (I - D)^{-1} \cdot \left[ P(q) \cdot q - C(q) \cdot 1 \right] \]

\[ = 1 \cdot \left[ P(q) \cdot q - C(q) \cdot 1 \right] \]

and the thesis holds. \( \square \)

This way, with Formulation 4, profits are no longer overcounted. While in giving their results both [8] and [4] use a formulation that is equivalent to Formulation 3 they use an argument similar to the second part of our proof to prove that profits are not overstated. This however does not approaches some other drawbacks of Formulation 3; they will be discussed in Section 4.

3. The Lerner index and cartelizing effects

The Lerner index is defined as the ratio between the profit margin and the price, see [16] for details. The Lerner index is called also relative markup and is used to analyze the cartelizing effects of interlocks, for examples see [7] and [10]. It is therefore important to derive an expression for it since different profit formulations may affect the magnitude of cartelizing effects.

When the profit vector has been solved for, it is possible to derive a general expression for the Lerner index.
Proposition 3. Let the profit vector be written in the general formulation (6):
\[ \pi = A \cdot [p(q) \cdot q - C(q) \cdot 1] \]
The Lerner index vector may be express in the following way:
\[ L(q) \cdot 1 = \frac{1}{\xi} (\text{diag } A)^{-1} \cdot A \cdot s \]
where:
- \( L(q) \) is the diagonal matrix where i-th diagonal element is Firm i’s Lerner index \( L_i \).
- \( s \) is the market share column vector.

Proof. Expanding the general formulation (6) we get:
\[
\begin{cases}
\pi_1 = \sum_{j=1}^{n} a_{1j}[p(q)q_j - c(q_j)] \\
\vdots \\
\pi_n = \sum_{j=1}^{n} a_{nj}[p(q)q_j - c(q_j)]
\end{cases}
\]
where \( a_{ij} \) is the generic \( A \) element. Considering FOC:
\[
\begin{cases}
a_{11}[p(q) - c'_1(q_1)] = - \sum_{j=1}^{n} a_{1j}p'(q)q_j \\
\vdots \\
a_{nn}[p(q) - c'_n(q_n)] = - \sum_{j=1}^{n} a_{nj}p'(q)q_j.
\end{cases}
\]
By simple algebra, since we assume \( \xi(q) \neq 0 \) at the equilibrium point:
\[
\begin{cases}
a_{11} \frac{p(q) - c'_1(q_1)}{p(q)} = - \sum_{j=1}^{n} a_{1j} \frac{p'(q)q_j}{p(q)} \frac{q_j}{q} \\
\vdots \\
a_{nn} \frac{p(q) - c'_n(q_n)}{p(q)} = - \sum_{j=1}^{n} a_{nj} \frac{p'(q)q_j}{p(q)} \frac{q_j}{q}.
\end{cases}
\]
This can be written:
\[
\begin{cases}
a_{11}L_1 = \frac{1}{\xi} \sum_{j=1}^{n} a_{1j}s_j \\
\vdots \\
a_{nn}L_n = \frac{1}{\xi} \sum_{j=1}^{n} a_{nj}s_j
\end{cases}
\]
that is:
\[
L(q) \cdot 1 = \frac{1}{\xi} (\text{diag } A)^{-1} \cdot A \cdot s.
\]

In the symmetrical case it is a simple matter of algebra to obtain the following expressions for each different formulation. In particular, it holds:

**Corollary 4.** In the symmetrical case, when considering Formulation 3 and 4, Firm i’s Lerner index is the same. In particular it is:

\[
L_i = \frac{1}{\xi} \left[ s_i + \sum_{k \neq i} \frac{\delta}{1 - \delta(n - 2)} s_k \right] = \frac{1}{\xi} \left[ s_i + \frac{\delta}{1 - \delta(n - 2)} (1 - s_i) \right].
\]

**Proof.** It is sufficient to prove that \((\text{diag } A)^{-1} \cdot A\) is the same, under both formulations, i.e., when \(A = (I - D)^{-1}\) (Formulation 3) and when \(A = \text{diag } [I - (1 \cdot D)] \cdot (I - D)^{-1}\) (Formulation 4). Since

\[
(I - D)^{-1} = \begin{pmatrix}
\frac{1}{(1 + \delta)(1 - n)\delta} & \frac{\delta}{1 - (n - 2)\delta} & \cdots & \frac{\delta}{1 - (n - 2)\delta} \\
\frac{\delta}{1 - (n - 2)\delta} & \frac{1}{(1 + \delta)(1 - n)\delta} & \cdots & \frac{\delta}{1 - (n - 2)\delta} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\delta}{1 - (n - 2)\delta} & \frac{\delta}{1 - (n - 2)\delta} & \cdots & \frac{1}{(1 + \delta)(1 - n)\delta}
\end{pmatrix}
\]

under Formulation 3 we have:

\[
(\text{diag } A)^{-1} \cdot A = \begin{pmatrix}
1 & \frac{\delta}{1 - (n - 2)\delta} & \cdots & \frac{\delta}{1 - (n - 2)\delta} \\
\frac{\delta}{1 - (n - 2)\delta} & 1 & \cdots & \frac{\delta}{1 - (n - 2)\delta} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\delta}{1 - (n - 2)\delta} & \frac{\delta}{1 - (n - 2)\delta} & \cdots & 1
\end{pmatrix}
\]

while, when considering Formulation 4, by simple algebra it is immediate to find

\[
\text{diag } [I - (1 \cdot D)] \cdot (I - D)^{-1}
\]

\[
= \begin{pmatrix}
\frac{1}{1 + \delta} & \frac{\delta}{1 + \delta} & \cdots & \frac{\delta}{1 + \delta} \\
\frac{\delta}{1 + \delta} & \frac{1}{1 + \delta} & \cdots & \frac{\delta}{1 + \delta} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\delta}{1 + \delta} & \frac{\delta}{1 + \delta} & \cdots & \frac{1}{1 + \delta}
\end{pmatrix}
\]
and also with this formulation

\[
(\text{diag } A)^{-1} \cdot A = \begin{pmatrix}
\frac{1}{\delta} & \frac{\delta}{1-(n-2)\delta} & \cdots & \frac{\delta}{1-(n-2)\delta} \\
\frac{\delta}{1-(n-2)\delta} & \frac{1}{\delta} & \cdots & \frac{\delta}{1-(n-2)\delta} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\delta}{1-(n-2)\delta} & \frac{\delta}{1-(n-2)\delta} & \cdots & \frac{1}{\delta}
\end{pmatrix}.
\]

Finally, by simple substitution, it is immediate to find Firm \( i \)’s Lerner index expression.

\[\square\]

While [14] compare Formulations 1 and 2 in terms of output, [7] compares Formulations 2 and 3 in terms of Lerner index and prove that “... the cartelizing effects of shareholding interlocks are compounded if firms consider also indirect shareholding.” The approach used in [7] is to consider symmetric shareholding, i.e., \( \delta_{ik} = \delta \), and consider the limit of the Lerner index as \( \delta \to (1/(n-1))^{-} \). It is evident that in this case, cartelizing effects are studied when firms have relatively large financial interests in each other. It must be noted that in the limit case each firm is totally controlled by the others. This is particular evident for Formulation 4 but it is true implicitly also for Formulation 3. Optimal quantities \( q^* \) depend on \( \delta \), but in the limit case any quantity is optimal; as a consequence \( \lim_{\delta \to 1/(n-1)} q^*(\delta) \) does not exist. So, since Lerner index depends on optimal shares which, on turn, depend on optimal quantities, the study of limit case must be interpreted with the utmost care. In facts the cartelizing effects are proved only when the situation is close to the total control case, but in this case the optimal quantities tends not to be defined. In the following section we examine a counterexample showing this point.

4. The linear case

In this section we consider Formulation 4 and study cartelizing effects when small financial interests are present. We will show how, even in a simple case, it is possible to observe that it is not always rational to extend financial control in each other as much as possible. Finally, our analysis challenges some of the results given in [7]: in particular, we prove that with the correct formulation, in the limit as \( \delta \) approaches its upper bound, the industry does not become a perfect cartel.

Consider two firms with linear inverse demand

\[ P = a - b(q_1 + q_2) \]
and linear costs

\[ c_i(q_i) = c_i q_i , \quad a > c_i , \quad i = 1, 2 . \]

It is possible to show that financial control does not always increase profit of firms.

**Theorem 5.** Assume no firm has financial interest in the competitor, and both are producing positive quantities, then

- **Firm 1** has interest in acquiring asymmetric financial control in **Firm 2** if and only if \( c_1 > (7c_2 - 2a)/5 \).
- **Small symmetric financial control** increases **Firm 1’s** profit only if \( c_1 \in [(13 + 3\sqrt{13})c_2 - (11 + 3\sqrt{13})a]/2 \) or \( c_1 > [(13 - 3\sqrt{13})c_2 - (11 - 3\sqrt{13})a]/2 \).
- **When financial control** is close to 1 both firms have incentive to increase shareholding interlocks.

**Proof.**

- Assume \( 0 \leq c_1 \leq a \) and \( 0 \leq c_2 \leq a \), reaction functions are:

\[
R_1(q_2) = \begin{cases} 
\frac{a-b(1+\delta_{12})q_2-c_1}{2b} & \text{if } a-(1+\delta_{12})b-c_1 > 0 \\
0 & \text{if } a-(1+\delta_{12})b-c_1 \leq 0 
\end{cases} \\
R_2(q_1) = \begin{cases} 
\frac{a-b(1+\delta_{21})q_1-c_2}{2b} & \text{if } a-(1+\delta_{21})b-c_2 > 0 \\
0 & \text{if } a-(1+\delta_{21})b-c_2 \leq 0 
\end{cases}
\]

As a consequence optimal quantities are:

\[
(q_1^*, q_2^*) = \begin{cases} 
(0, \frac{a-c_2}{2b}) & \text{if } (1+\delta_{12})a-2c_1+(1+\delta_{12})c_2 < 0 \\
\left(\frac{a-c_1}{2b}, 0\right) & \text{if } (1+\delta_{21})a-2c_2+(1+\delta_{21})c_1 < 0 \\
\left(\frac{(1+\delta_{12})a-2c_1+(1+\delta_{12})c_2}{3-\delta_{12}-\delta_{21}-\delta_{12}\delta_{21}}, \frac{(1+\delta_{21})a-2c_2+(1+\delta_{21})c_1}{3-\delta_{12}-\delta_{21}-\delta_{12}\delta_{21}}\right) & \text{elsewhere} .
\end{cases}
\]

Recalling that profit for **Firm 1** is

\[
\pi_1(q_1, q_2) = \frac{(1+\delta_{21})((a-b(q_1+q_2))q_1-c_1q_1)}{(1-\delta_{12}\delta_{21})} + \frac{\delta_{12}(a-b(q_1+q_2))q_2-c_2q_2}{1-\delta_{12}\delta_{21}}.
\]
When $\delta_{21}$ is identically null, and both optimal quantities are positive, Firm 1 profit depends on $\delta_{12}$. By simple substitution we obtain

$$
\frac{\partial \pi_1}{\partial \delta_{12}} = \frac{\left(2a^2 + 7ac_1 + 5c_1^2 - 11ac_2 - 17c_1c_2 + 14c_1^2 + ac_1\delta_{12}\right)}{(3 - \delta_{12})^3b}.
$$

When $\delta_{12}$ is small:

$$
\frac{\partial \pi_1}{\partial \delta_{12}} \sim \frac{2a^2 + 7ac_1 + 5c_1^2 - 11ac_2 - 17c_1c_2 + 14c_1^2}{27b}.
$$

While this derivative is positive if and only if $c_1 < 2c_2 - a$ or $c_1 > (7c_2 - 2a)/5$, it must be noted that the first condition is equivalent to $(1 - \delta_{21})a - 2c_2 + (1 + \delta_{21})c_1 < 0$ when $\delta_{21} = 0$, as a consequence it is of no interest.

- In the case of symmetric control the derivative of Firm 1 profit may be written

$$
\frac{\partial \pi_1}{\partial \delta} = \frac{\alpha_0 \delta^5 + \alpha_1 \delta^4 + \alpha_2 \delta^3 + \alpha_3 \delta^2 + \alpha_4 \delta + \alpha_5}{(1 - \delta)^2(1 + \delta)^2(3 + \delta)^3b}
$$

where:

- $\alpha_0 = -a^2 + ac_1 + ac_2 - c_1c_2$,
- $\alpha_1 = a^2 + ac_1 + 2c_1^2 - 3ac_2 - 5c_1c_2 + 4c_2^2$,
- $\alpha_2 = 2a^2 + 4ac_1 + 9c_1^2 - 8ac_2 - 22c_1c_2 + 15c_2^2$,
- $\alpha_3 = -2a^2 - 4ac_1 + 25c_1^2 + 8ac_2 - 46c_1c_2 + 19c_2^2$,
- $\alpha_4 = -a^2 - 13ac_1 + 27c_1^2 + 15ac_2 - 41c_1c_2 + 13c_2^2$,
- $\alpha_5 = a^2 + 11ac_1 + c_1^2 - 13ac_2 - 13c_1c_2 + 13c_2^2$.

- When $\delta$ is small:

$$
\frac{\partial \pi_1}{\partial \delta} \sim \frac{a^2 + 11ac_1 + c_1^2 - 13ac_2 - 13c_1c_2 + 13c_2^2}{27b}.
$$

The thesis follows.

- When $\delta \to 1^-$ we have $\frac{\partial \pi_1}{\partial \delta} \sim \frac{(8c_1 - 8c_2)^2}{0^+} > 0$ it follows that there is incentive to increase $\delta$. \[\square\]

**Remark.** In the case of same cost for both firms it is immediate to prove that Firm 1 profit is always increasing since, for $c < a$, it holds $c > (7c - 2a)/5$. 

This result is extremely important because it considers marginal costs. In the case of asymmetrical control profit increases when controlling a firm with lower marginal cost while when considering symmetric control there is a trade-off in between investing in a higher marginal cost and cartelizing effects.

**Corollary 6.** In the case of symmetrical control assume no firm has financial interest in the competitor. Then, for at least one firm it is always optimal to increase shareholding interlocks.

**Proof.** Recall $a > c_1$ and $a > c_2$. In particular this implies:

$$\frac{(13 + 3\sqrt{13})c_2 - (11 + 3\sqrt{13})a}{2} < \frac{(13 - 3\sqrt{13})c_2 - (11 - 3\sqrt{13})a}{2} < c_2$$

and a similar inequality holds for Firm 2.

By contradiction assume no firm has incentive in having symmetrical shareholding interlocks:

$$\begin{cases} \frac{\partial \pi_1}{\partial \delta} < 0, \\ \frac{\partial \pi_2}{\partial \delta} < 0. \end{cases}$$

Then a necessary condition is

$$\begin{cases} c_1 < \frac{(13 - 3\sqrt{13})c_2 - (11 - 3\sqrt{13})a}{2}, \\ c_2 < \frac{(13 - 3\sqrt{13})c_1 - (11 - 3\sqrt{13})a}{2}. \end{cases}$$

but this holds if and only if $c_1 \geq a \land c_2 \geq a$. Clearly absurd. \(\square\)

These results prove that the cartelizing effects of cross-shareholding depend also on the cost function of the firms, and not just on the financial structure of the firms. Furthermore they can explain the empirical evidence where cross-shareholding interlocks are non-symmetrical and limited (see for example [4]). In fact the examples in the linear case show that, in some cases, there may be no incentive to have financial control in rivals.

Finally, as in the theoretical literature, it is possible to study situations where the financial control is large, i.e., $(\delta_{12}, \delta_{21})$ is close to (1, 1).
Theorem 7. Consider two interlocked firms. In the linear case the following hold:

- in the limit case of shareholding interlocks Firm i’s optimal quantity does not exist, while the limit of total quantity exists only when firms have the same marginal cost. In this case the limit of total output is the same of the monopoly
- the limit of Firm i’s net profit does not exist nevertheless the limit of aggregate profit is formally equal to the sum of the operative earnings
- when Firm 2 is a division of Firm 1, i.e., $\delta_{12} = 1$ and $\delta_{21} = 0$, the total quantity is the monopoly quantity.

Proof.

- Consider $q^*_1$, by polar coordinates it is simple to obtain:
  \[
  \lim_{(\delta_{12}, \delta_{21}) \to (1,1)} q^*_1 = \lim_{\rho \to 0} \frac{\rho \cos \theta (a - c_2) a + 2c_2 - c_1)}{\rho(2 \cos \theta + 2 \sin \theta - \rho \cos \theta \sin \theta)b}
  = \begin{cases} 
  +\infty & \text{if } c_1 < c_2, \\
  -\infty & \text{if } c_1 > c_2, \\
  \frac{(a - c) \cos \theta}{2b(\sin \theta + \cos \theta)} & \text{if } c_1 = c_2 = c.
  \end{cases}
  \]

  As it concerns $q^*_2$:
  \[
  \lim_{(\delta_{12}, \delta_{21}) \to (1,1)} q^*_2 = \begin{cases} 
  +\infty & \text{if } c_1 < c_2, \\
  -\infty & \text{if } c_1 > c_2, \\
  \frac{(a - c) \sin \theta}{2b(\sin \theta + \cos \theta)} & \text{if } c_1 = c_2 = c.
  \end{cases}
  \]

  Finally:
  \[
  \lim_{(\delta_{12}, \delta_{21}) \to (1,1)} q^*_1 + q^*_2 = \begin{cases} 
  \frac{(a - c_1) \sin \theta + (a - c_2) \cos \theta}{2b(\sin \theta + \cos \theta)} & \text{if } c_1 \neq c_2, \\
  \frac{a - c}{2b} & \text{if } c_1 = c_2.
  \end{cases}
  \]

- Analogously it is easy to prove:
  \[
  \lim_{(\delta_{12}, \delta_{21}) \to (1,1)} \pi_1 = \frac{\sin \theta}{\sin \theta + \cos \theta}(OE_1^* + OE_2^*),
  \]
  \[
  \lim_{(\delta_{12}, \delta_{21}) \to (1,1)} \pi_2 = \frac{\cos \theta}{\sin \theta + \cos \theta}(OE_1^* + OE_2^*),
  \]
  where $OE_i^*$ is Firm i’s operative earning when $q_1 = q^*_1$ and $q_1 = q^*_2$.

  Obviously the thesis holds only formally since, in the limit case, nor optimal quantities nor their limit exist.
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By simple substitution:

\[ q_1^* + q_2^* = \frac{a - 2c_1 + 2c_2}{2b} + \frac{a - 2c_2 + c_1}{2b} = \frac{a - c_1}{2b}. \]

In the last case total quantity is the same as if Firm 1 were a monopolist. Furthermore when Firm 1 has complete financial control of Firm 2 it is reasonable to assume that \( q_1 \) and \( q_2 \) are both chosen by Firm 1:

\[ \max_{q_1, q_2} [p(q_1 + q_2) - c_1]q_1 + [p(q_1 + q_2) - c_2]q_2. \]

When \( \delta_{12} = 1 \) and \( \delta_{21} = 0 \) this is equivalent to:

\[ \max \pi_1. \]

Finally, as mentioned in Section 3, [7], considers the Lerner index to prove that in the symmetric case the industry becomes a perfect cartel. Since the used technique consists of studying the limit of the Lerner index as \( \delta \to 1/(n - 1) \) our analysis shows that, even in the linear case with two firm (\( n = 2 \)), this result must be taken with the utmost care, since optimal quantities do not always exist.

5. Oligopoly with product differentiation

It is interesting to study and compare these different profit formulations in the case of differentiated product. Consider an industry with \( n \) producers of non-homogeneous products. Each firm produces just one good. Because of product differentiation it is assumed that the unit price \( p_i \) of Firm \( i \)'s good depends on the production level of each of the firms: \( P_i(q_1, q_2, \ldots, q_n) \). In this model firms are still in a static Cournot-Nash equilibrium with choices of quantities representing strategies. (For the case of multiproduct monopoly the reader may refer to [16], and, in this particular setting, to [10]).

The notation is modified as follows:

- \( P(q) \): diagonal matrix where \( i \)-th diagonal element is good \( i \)'s inverse demand function.

The formulations are analogous to the homogenous product case and may assume a general matricial formulation. This way it is possible to compare the various formulations via matrix notation. In fact, provided matrix \( A \) is defined, the vector of profits can be expressed as:

\[ \pi = A \cdot [P(q) \cdot q - C(q) \cdot 1]. \]
The different formulations simply differ in the choice of the matrix $A$ which is the same as the one reported for the homogeneous product.

In the product differentiation case the derivation of the Lerner index is analogous to the homogenous product case.

**Proposition 8.** Consider the profit vector written in the general formulation (8). The Lerner index vector may be expressed in the following way

$$L(q) \cdot 1 = (\text{diag } A)^{-1} \cdot (R)^{-1} \left[ \left( \frac{1}{\epsilon} \right)^T \otimes_e A \right] \cdot R \cdot 1 \quad (9)$$

where

- $R$ is the diagonal matrix where $i$-th diagonal element is Firm $i$'s revenue $R_i := q_i p_i$,
- $1/\epsilon$ is the matrix with elements the reciprocal of $\epsilon_{ij}$, the inverse demand partial elasticity,
- $\otimes_e$ denotes elementwise matrix multiplication.

**Proof.** Consider profit formulation in multiproduct:

$$\begin{align*}
\pi_1 &= \sum_{j=1}^{n} a_{1j} [p_j(q)q_j - c_j(q_j)]q_j \\
\pi_2 &= \sum_{j=1}^{n} a_{2j} [p_j(q)q_j - c_j(q_j)]q_j \\
&\vdots \\
\pi_n &= \sum_{j=1}^{n} a_{nj} [p_j(q)q_j - c_j(q_j)]q_j.
\end{align*}$$

Rearrange FOCs:

$$\begin{align*}
a_{11}[p_1(q) - c_1'(q_1)] &= - \sum_{j=1}^{n} a_{1j} \frac{\partial p_j}{\partial q_1} q_j \\
a_{22}[p_2(q) - c_2'(q_2)] &= - \sum_{j=1}^{n} a_{2j} \frac{\partial p_j}{\partial q_1} q_j \\
&\vdots \\
a_{nn}[p_n(q) - c_n'(q_n)] &= - \sum_{j=1}^{n} a_{nj} \frac{\partial p_j}{\partial q_n} q_j.
\end{align*}$$
It is a matter of simple algebra to obtain:

\[
\begin{bmatrix}
L_1 \\
L_2 \\
\vdots \\
L_n
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{a_{11} R_1} \sum_{j=1}^{n} \frac{a_{1j} R_j}{\epsilon_{j1}} \\
\frac{1}{a_{22} R_2} \sum_{j=1}^{n} \frac{a_{2j} R_j}{\epsilon_{j2}} \\
\vdots \\
\frac{1}{a_{nn} R_n} \sum_{j=1}^{n} \frac{a_{nj} R_j}{\epsilon_{jn}}
\end{bmatrix}.
\] 

(10)

Finally, expanding expression (9), one obtains (10) and this proves the thesis.

\[\Box\]

6. Conclusion

The literature about financial interlocks proposes various profit formulations. Cartelizing effects have been established theoretically, yet, according to [6], empirical examples of silent partial equity interests among rivals are almost nonexistent. In this paper we examined and discussed the different profit formulations used to study cartelizing effects of shareholding and, finally, we proposed a formulation which corrects the overcounting profits problem. In particular, with this formulation the industry aggregate net profit amounts to the industry aggregate operative earnings. Using matrix notation, we gave all formulations a common structure, thereby deriving more easily the expression of Lerner index and allowing extension to consider product differentiation. The analysis of the mathematical details overlooked in previous literature allowed us to explain some inconsistencies and give an explanation of the differences between the empirical evidence and theoretical results predictions. By a simple counterexample we showed that, in contrast to some of the results given by [7], in the limit, as \(\delta\) approaches its upper bound, it is not possible to claim that the industry does become a perfect cartel. Finally it must be noted how the mathematical details considered in our analysis are not just technicalities but have an economic interpretation; considering them properly allowed a deeper understanding of the model and simpler explanation of the empirical evidence.

The profit formulation we suggest, together with the extension to the product differentiation, may be considered a more solid theoretical model in order to perform empirical analysis of cross-shareholdings.
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